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Stability Bounds for Local Lagrangian Interpolation

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1. INTRODUCTION

The problem studied in this paper is the analysis of the stability bounds for local Lagrangian (polynomial) interpolation as a function of mesh ratios. The results are useful in studying the behavior of local piecewise polynomial interpolation on highly non-uniform meshes [3]. Swartz and Varga [5], in their extensive study of stability, consider only quasi-uniform meshes and the effect of the mesh ratio is absorbed into generic constants. Also, their results do not consider the case of multiple interpolation points as is done here. Prenter [2] has a result similar to that of (3.6) below, but again only distinct interpolation points are considered. This paper presents stability bounds for both the cases of quasi-uniform meshes and locally quasi-uniform meshes where the dependence on the mesh ratio is explicitly given.

Let $\{x_n\}_{n=1}^d$ in $[a, b]$ be given with respective multiplicities $\{\mu_n\}_{n=1}^d$. Set

$$N = \sum_{n=1}^{d} \mu_n$$

and let $\{\phi_{rs}\}$, *the Lagrange polynomials*, be those functions in $\mathbf{P}_N$, the space of polynomials of degrees *less than N, which satisfy*

$$\phi_{rs}^{(j-1)}(x_i) = \delta_{ir}\,\delta_{js} \qquad 1 \leqslant j \leqslant \mu_i, \quad 1 \leqslant i \leqslant d, \tag{1.1}$$

for $1 \leqslant s \leqslant \mu_r$, $1 \leqslant r \leqslant d$. Then for any $f \in C^N[a, b]$

$$(Qf)(x) := \sum_{r=1}^{d} \sum_{s=1}^{\mu_r} f^{(s-1)}(x_r)\,\phi_{rs}(x) \tag{1.2}$$

satisfies the generalized interpolation conditions

$$(Qf^{(j-1)})(x_i) = f^{(j-1)}(x_i) \qquad 1 \leqslant j \leqslant \mu_i, \quad 1 \leqslant i \leqslant d.$$

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To study the stability of this interpolation, it is necessary to examine

$$\sum_{r=1}^{d} \sum_{s=1}^{\mu_r} \|\phi_{rs}\|$$

(here $\|\cdot\|$ is the maximum norm over $[a, b]$). In fact, bounds will be established for $|\phi_{rs}^{(j-1)}(x)|$ when $x \in [x_n, x_{n+1}]$ for general n, j, r, s.

An argument reminiscent of [1, pp. 289–290] establishes that the error in estimating f by Qf satisfies

$$\|(f-Qf)^{(j-1)}\| = O(h^{N+1-j}) \qquad 1 \leqslant j \leqslant N, \tag{1.3}$$

for $f \in C^N[a, b]$. Here, and in what follows, $h_i := x_{i+1} - x_i$ for $1 \leqslant i < d$, and $h := \max_i h_i$. For example, by repeated applications of Rolle's theorem there is a z in (x_1, x_d) such that $(f-Qf)^{(N-1)}(z) = 0$; hence, $(f-Qf)^{(N-1)}(x) = (x-z) f^{(N)}(\xi)$ for some ξ in (x_1, x_d), i.e., $\|(f-Qf)^{(N-1)}\| \leqslant (d-1)\, h\, \|f^{(N)}\|$. Note that the order constant in (1.3) does not depend on the distribution of the mesh $\{x_i\}$. However, if the quantities $f^{(s-1)}(x_r)$ are replaced by $O(\varepsilon)$-accurate estimates $\hat{f}^{(s-1)}(x_r)$, then from (1.2)

$$|(Qf - Q\hat{f})^{(j-1)}| \leqslant \sum_r \sum_s |\phi_{rs}^{(j-1)}| \cdot O(\varepsilon), \tag{1.4}$$

so stability depends on the bounds for $|\phi_{rs}^{(j-1)}|$. As will be shown, these bounds can be quite large for highly non-uniform meshes.

Two classes of meshes are considered: quasi-uniform and locally quasi-uniform. A quasi-uniform mesh, with mesh ratio σ, is one for which

$$1/\sigma \leqslant h_i/h_j \leqslant \sigma \qquad \text{for every } i, j. \tag{1.5}$$

A locally quasi-uniform mesh, with local mesh ratio R, is one for which

$$1/R \leqslant h_{i+1}/h_i \leqslant R \qquad \text{for every } i. \tag{1.6}$$

For the latter class of meshes, the analog of (1.5) is

$$1/R^{|i-j|} \leqslant h_i/h_j \leqslant R^{|i-j|} \qquad \text{for every } i, j. \tag{1.7}$$

In what follows C represents a positive generic constant, possibly dependent on $\{\mu_n\}$, d, and $b-a$. However, it will not depend on σ or R, as the explicit dependence of stability bounds on mesh ratios is the main object of this paper.

The Lagrange polynomials ϕ_{rs} can be generated recursively, as is well known (e.g., [4, p. 53]). Define the auxiliary Lagrange polynomials $L_{rs}(x)$ by

$$L_{rs}(x) := \frac{(x-x_r)^{s-1}}{(s-1)!} \prod_{\substack{j=1 \\ j \neq r}}^{d} \left(\frac{x-x_j}{x_r-x_j}\right)^{\mu_j} \tag{1.8}$$

for $1 \leqslant s \leqslant \mu_r$, $1 \leqslant r \leqslant d$. Then $\phi_{r,\mu_r}(x) = L_{r,\mu_r}(x)$ and for $1 \leqslant s < \mu_r$,

$$\phi_{rs}(x) = L_{rs}(x) - \sum_{p=s+1}^{\mu_r} L_{rs}^{(p-1)}(x_r)\, \phi_{rp}(x),$$

from which $\phi_{r,\mu_r}^{(j-1)} = L_{r,\mu_r}^{(j-1)}$, and for $1 \leqslant s < \mu_r$,

$$\phi_{rs}^{(j-1)}(x) = L_{rs}^{(j-1)}(x) - \sum_{p=s+1}^{\mu_r} L_{rs}^{(p-1)}(x_r)\, \phi_{rp}^{(j-1)}(x). \tag{1.9}$$

Clearly, it is necessary to study $L_{rs}(x)$, so first consider (1.8) for general meshes. It is convenient to write $L_{rs}(x) = \prod_{j=1}^{d} F_j(x)$, where $F_j(x) := [(x-x_j)/(x_r-x_j)]^{\mu_j}$ for $j \neq r$, and $F_r(x) := (x-x_r)^{s-1}/(s-1)!$. (While it is true that F_j also depends on r and s, these subscripts have been omitted for clarity.) A simple extension of Leibnitz' rule for products yields

$$\begin{aligned}(F_1 F_2 \cdots F_d)^{(k_0-1)} = &\sum_{k_1=1}^{k_0} \sum_{k_2=1}^{k_1} \cdots \sum_{k_{d-1}=1}^{k_{d-2}} \binom{k_0-1}{k_1-1}\binom{k_1-1}{k_2-1} \cdots \binom{k_{d-2}-1}{k_{d-1}-1} \\ &\times F_1^{(k_0-k_1)} F_2^{(k_1-k_2)} \cdots F_d^{(k_{d-1}-k_d)},\end{aligned}$$

where $k_d := 1$. As a straightforward consequence of this expansion, we have equations for $L_{rs}^{(k_0-1)}(x)$ and $L_{rs}^{(k_0-1)}(x_r)$.

LEMMA 1. *For $k_0 - 1 \leqslant$ degree L_{rs},*

$$\begin{aligned}L_{rs}^{(k_0-1)}(x) = &\sum_{k_1} \cdots \sum_{k_{d-1}} (x-x_r)^{s-1-k_{r-1}+k_r} \prod_{\substack{j=1 \\ j \neq r}}^{d} \\ &\times \frac{(x-x_j)^{\mu_j - k_{j-1}+k_j}\, C(k_0, ..., k_d, \mu_j, s)}{(x_r-x_j)^{\mu_j}},\end{aligned} \tag{1.10}$$

where $0 \leqslant k_{j-1} - k_j \leqslant \mu_j$ for $j \neq r$, and $0 \leqslant k_{r-1} - k_r \leqslant s-1$. Also,

$$L_{rs}^{(k_0-1)}(x_r) = \sum_{k_1} \cdots \sum_{k_{d-1}} \prod_{\substack{j=1 \\ j \neq r}}^{d} \frac{C(k_0, ..., k_d, \mu_j, s)}{(x_r-x_j)^{k_{j-1}-k_j}}, \tag{1.11}$$

where $0 \leqslant k_{j-1} - k_j \leqslant \mu_j$ for $j \neq r$, and $k_{r-1} - k_r = s-1$.

2. Quasi-Uniform Meshes

Assume that $\{x_i\}$ satisfies (1.5) for some $\sigma > 0$; we seek bounds for $|\phi_{rs}^{(j-1)}|$ explicitly exhibiting the dependence on σ. These bounds will be over a fixed reference interval $[x_n, x_{n+1}]$ so h_n factors will appear.

To bound $L_{rs}^{(k_0-1)}$ in terms of σ and h_n, it is first necessary to study the behavior of the individual factors in (1.10)–(1.11). For $x \in [x_n, x_{n+1}]$ in case $j \leqslant n$ and $r \leqslant n$

$$\begin{aligned} \left|\frac{(x-x_j)^{\mu_j - k_{j-1} + k_j}}{(x_r - x_j)^{\mu_j}}\right| &\leqslant \frac{(x_{n+1} - x_j)^{\mu_j - k_{j-1} + k_j}}{|x_r - x_j|^{\mu_j}} \\ &\leqslant C \max\left\{\frac{(\sigma h_n)^{\mu_j - k_{j-1} + k_j}}{h_n^{\mu_j}}, \frac{h_n^{\mu_j - k_{j-1} + k_j}}{(h_n/\sigma)^{\mu_j}}\right\} \\ &= C\sigma^{\mu_j}/h_n^{k_{j-1} - k_j}. \end{aligned}$$

Similarly, for $j > n$ and $r > n$ the bound is also $C\sigma^{\mu_j}/h_n^{k_{j-1} - k_j}$, while for $r \leqslant n < j$ or $j \leqslant n < r$ the bound is $1/h_n^{k_{j-1} - k_j}$.

Lemma 2. *For $k_0 - 1 \leqslant$ degree L_{rs} and x in $[x_n, x_{n+1}]$*

$$|L_{rs}^{(k_0-1)}(x)| \leqslant C\sigma^{G(r,n) - \mu_r}/h_n^{k_0 - s}, \tag{2.1}$$

where

$$G(r, n) = \begin{cases} \sum_{j=1}^{n} \mu_j & r \leqslant n \\ \sum_{j=n+1}^{d} \mu_j & r > n; \end{cases}$$

$$|L_{rs}^{(k_0-1)}(x_r)| \leqslant C(\sigma/h_n)^{k_0 - \varepsilon}. \tag{2.2}$$

Proof. From Lemma 1, when $r \leqslant n$

$$\begin{aligned} |L_{rs}^{(k_0-1)}(x)| \leqslant C \sum_{k_1} \cdots \sum_{k_{d-1}} \prod_{j=1}^{r-1} & \frac{(x_{n+1} - x_j)^{\mu_j - k_{j-1} + k_j}}{(x_r - x_j)^{\mu_j}} \\ & \cdot \prod_{j=r+1}^{n} \frac{(x_{n+1} - x_j)^{\mu_j - k_{j-1} + k_j}}{(x_j - x_r)^{\mu_j}} \\ & \times \frac{h_n^{\mu_{n+1} - k_n + k_{n+1}}}{(x_{n+1} - x_r)^{\mu_{n+1} + k_{r-1} - k_r + 1 - s}} \cdot \prod_{j=n+2}^{d} \frac{(x_j - x_n)^{\mu_j - k_{j-1} + k_j}}{(x_j - x_r)^{\mu_j}} \\ \leqslant C \sum \cdots \sum & \sigma^{\sum_{j=1}^{r-1} \mu_j + \sum_{j=r+1}^{n} \mu_j}/h_n^e, \end{aligned} \tag{2.3}$$

where $e = (k_0 - k_{r-1}) + (k_r - k_n) + (k_{r-1} - k_r + 1 - s + k_n - k_{n+1}) + (k_{n+1} - k_d) = k_0 - s$ (recall $k_d = 1$). A similar argument works for $r > n$. As for (2.2), $1/|x_r - x_j|^{k_{j-1} - k_j} \leqslant (\sigma/h_n)^{k_{j-1} - k_j}$; similarly from (1.11),

$$|L_{rs}^{(k_0-1)}(x_r)| \leqslant C \sum \cdots \sum (\sigma/h_n)^{\sum_{j \neq r}(k_{j-1} - k_j)} \leqslant C(\sigma/h_n)^{k_0 - k_{r-1} + k_r - k_d}.$$

But $k_0 - k_{r-1} + k_r - k_d = k_0 + 1 - s - 1 = k_0 - s$. ∎

The bound on the individual Lagrange functions ϕ_{rs} can now be established from the recursion (1.9).

THEOREM 1. *For $j \leqslant N$ and x in $[x_n, x_{n+1}]$*

$$|\phi_{rs}^{(j-1)}(x)| \leqslant \begin{cases} Ch_n^{s-j} & 1 = r = n \quad \text{or} \quad n+1 = r = d \\ Ch_n^{s-j} \sigma^{G(r,n)-s}, & \text{otherwise}, \end{cases} \tag{2.4}$$

with $G(r, n)$ as in Lemma 2.

Proof. The argument is by induction on s (from μ_r down to 1). For $s = \mu_r$ we have $\phi_{r,\mu_r}(x) = L_{r,\mu_r}(x)$ and the result follows from Lemma 2. Assume true for $s \geqslant k+1$, then

$$\phi_{rk}^{(j-1)}(x) = L_{rk}^{(j-1)}(x) - \sum_{p=k+1}^{\mu_r} L_{rk}^{(p-1)}(x_r)\, \phi_{rp}^{(j-1)}(x).$$

Hence, in general,

$$\begin{aligned} |\phi_{rk}^{(j-1)}(x)| &\leqslant Ch_n^{k-j} \sigma^{G(r,n) - \mu_r} + C \sum_{p=k+1}^{\mu_r} (\sigma/h_n)^{p-k}\, h_n^{p-j} \sigma^{G(r,n)-p} \\ &\leqslant Ch_n^{k-j} \sigma^{G(r,n)-e}, \end{aligned}$$

where $e = \min(\mu_r, k) = k$, as desired. There are special cases when $r = n = 1$ or $r = n+1 = d$; e.g., $G(1, 1) = \mu_1$ so $|L_{1s}^{(k_0-1)}(x)| \leqslant C/h_1^{k_0-s}$ on $[x_1, x_2]$. Similarly, from (1.11),

$$|L_{1s}^{(k_0-1)}(x_1)| \leqslant C \sum \cdots \sum \prod_{j=2}^{d} \frac{1}{(x_j - x_1)^{k_{j-1} - k_j}} \leqslant C/h_1^{k_0 - s}.$$

Since these bounds are independent of σ, so is the one for $|\phi_{1s}^{(j-1)}(x)|$ on $[x_1, x_2]$. ∎

From Theorem 1 it is a simple manner to derive the stability results sought; in particular, with $e := \max(\sum_{p=1}^{n} \mu_p, \sum_{p=n+1}^{d} \mu_p)$,

$$\sum_{r=1}^{d} \sum_{s=1}^{\mu_r} |\phi_{rs}^{(j-1)}(x)| \leqslant Ch_n^{1-j} \sigma^{e-1} \tag{2.5}$$

for $x_n \leqslant x \leqslant x_{n+1}$ and $h_n \leqslant 1$. If $d > 2$ and the multiplicity is constant this becomes

$$\sum_{r=1}^{d} \sum_{s=1}^{\mu} |\phi_{rs}^{(j-1)}(x)| \leqslant Ch_n^{1-j} \sigma^{\mu \cdot \max(n, d-n)-1}. \tag{2.6}$$

As yet, no claim has been made as to the sharpness of the bounds in Lemma 2 or Theorem 1. The bound (2.1) on $L_{rs}^{(k_0-1)}$ can be shown sharp by considering the mesh

$$\begin{aligned} x_i &= (i-n)\, h_n/\sigma, & 1 \leqslant i \leqslant n \\ x_{i+1} &= h_n[1 + (i-n)/\sigma], & n \leqslant i \leqslant d-1, \end{aligned} \tag{2.7}$$

for which $[x_n, x_{n+1}] = [0, h_n]$. Let $v := \sum_{j \neq r} \mu_j + s - 1$, the degree of L_{rs}; then from (1.8)

$$L_{rs}^{(v)}(x) = \frac{v!}{(s-1)!} \prod_{j \neq r} (x_r - x_j)^{-\mu_j},$$

so for $r \leqslant n$,

$$L_{rs}^{(v)}(x) = \frac{v!}{(s-1)!} \prod_{j=1}^{r-1} \left[\frac{\sigma}{(r-j)\, h_n}\right]^{\mu_j} \prod_{j=r+1}^{n} \left[\frac{-\sigma}{(j-r)\, h_n}\right]^{\mu_j} \prod_{j=n+1}^{d} \times \left[\frac{-1}{[1 + (j-1-r)/\sigma]\, h_n}\right]^{\mu_j}$$

Hence,

$$\begin{aligned} |L_{rs}^{(v)}(x)| &= C\sigma^{G(r,n)-\mu_r} \prod_{j=n+1}^{d} \left(1 + \frac{j-1-r}{\sigma}\right)^{-\mu_j} \Big/ h_n^{\sum_{j \neq r} \mu_j} \\ &\geqslant C\sigma^{G(r,n)-\mu_r} / h_n^{v+1-s}, \end{aligned}$$

since $\sigma \geqslant 1$. A similar argument suffices for $r > n$. Finally, from Markov's inequality transformed to $[0, h_n]$,

$$\max_{[0, h_n]} |L_{rs}^{(j-1)}| \geqslant \frac{h_n}{2(v-j)^2} |L_{rs}^{(j)}(x)|$$

for any x in $[0, h_n]$ with $1 \leqslant j \leqslant v$. Thus,

$$\max_{[0, h_n]} |L_{rs}^{(j-1)}| \geqslant C\sigma^{G(r,n)-\mu_r} / h_n^{j-s} \tag{2.8}$$

by induction. An analogous argument will not work on $\phi_{rs}^{(j-1)}$ because of the possibility of cancellation in the recursion (1.9). In numerical experiments, however, the stability bound (2.6) has been shown to be sharp for meshes (2.7) even though (2.4) with this mesh is sometimes too generous for particular values of r and s.

3. Locally Quasi-Uniform Meshes

Assume that $\{x_i\}$ satisfies (1.6) for some $R>0$. The results in this section parallel those of the previous one so arguments will only be sketched. As before, the individual factors in (1.10)–(1.11) must be bounded. Six cases result depending on the relative ordering of r, n, and j. For $x \in [x_n, x_{n+1}]$, if $j < r \leqslant n$,

$$\left|\frac{(x-x_j)^{\mu_j - k_{j-1} + k_j}}{(x_r - x_j)^{\mu_j}}\right| \leqslant \frac{(x_{n+1}-x_j)^{\mu_j - k_{j-1}+k_j}}{(x_r - x_j)^{\mu_j}}$$

$$= \frac{(h_j + h_{j+1} + \cdots + h_n)^{\mu_j - k_{j-1} + k_j}}{(h_j + \cdots + h_{r-1})^{\mu_j}}$$

$$\leqslant CR^{(n-r+1)\mu_j}/h_n^{k_{j-1}-k_j}.$$

Similarly, for $r < j \leqslant n$, the bound is $CR^{(n-j+1)\mu_j}/h_n^{k_{j-1}-k_j}$, while for $r \leqslant n < j$ or $j \leqslant n < r$ the bound is $1/h_n^{k_{j-1}-k_j}$. Also, $h_n^{\mu_{n+1}-k_n+k_{n+1}}/(x_{n+1}-x_r)^{\mu_{n+1}+k_{r-1}-k_r+1-s} \leqslant h_n^{s-1+k_r-k_{r-1}+k_{n+1}-k_n}$, so substitution into (2.3) produces the first result.

Lemma 3. *For $k_0 - 1 \leqslant$ degree L_{rs} and x in $[x_n, x_{n+1}]$*

$$|L_{rs}^{(k_0-1)}(x)| \leqslant CR^{F(r,n)}/h_n^{k_0-s}, \tag{3.1}$$

where

$$F(r,n) = \begin{cases} (n-r+1)\displaystyle\sum_{j=1}^{r-1} \mu_j + \sum_{j=r+1}^{n} (n-j+1)\,\mu_j & r \leqslant n \\ \displaystyle\sum_{j=n+1}^{r-1} (j-n)\,\mu_j + (r-n)\sum_{j=r+1}^{d} \mu_j & r > n. \end{cases}$$

$$|L_{rs}^{(k_0-1)}(x_r)| \leqslant CR^{(k_0-s)e}/h_n^{k_0-s}, \tag{3.2}$$

where

$$e=\begin{cases} n-1 & 1=r\leqslant n \\ n-r+1 & 1<r\leqslant n \\ r-n & n<r<d \\ d-n-1 & n<r=d. \end{cases}$$

Proof. (3.1) follows directly from (2.3) and the above remarks. As for (3.2), from Lemma 1 when $n<r<d$,

$$\begin{aligned} &|L_{rs}^{(k_0-1)}(x_r)| \\ &\leqslant C\sum\cdots\sum\prod_{j=1}^{n}(x_r-x_j)^{k_j-k_{j-1}}\prod_{j=n+1}^{r-1}(x_r-x_j)^{k_j-k_{j-1}}\prod_{j=r+1}^{d}(x_j-x_r)^{k_j-k_{j-1}} \\ &\leqslant C\sum\cdots\sum R^{\sum_{j=n+1}^{r-1}(j-n)(k_{j-1}-k_j)+\sum_{j=r+1}^{d}(r-n)(k_{j-1}-k_j)}/h_n^{k_0-k_{r-1}+k_r-k_d}. \end{aligned}$$

But, $k_0-k_{r-1}+k_r-k_d=k_0+1-s-1=k_0-s$ and, summing by parts,

$$\begin{aligned} &\sum_{j=n+1}^{r-1}(j-n)(k_{j-1}-k_j)+\sum_{j=r+1}^{d}(r-n)(k_{j-1}-k_j) \\ &\qquad=\sum_{j=n}^{r-1}k_j+(r-n)(k_r-k_{r-1}-k_d) \\ &\qquad\leqslant(r-n)\,k_0+(r-n)(1-s-1)=(r-n)(k_0-s). \end{aligned}$$

The remaining cases follow similarly. ∎

The main result on the bounds of ϕ_{rs} follows directly from the recursion (1.9).

THEOREM 2. *For $j\leqslant N$ and x in $[x_n, x_{n+1}]$,*

$$|\phi_{rs}^{(j-1)}(x)|\leqslant Ch_n^{s-j}R^{E(r,n,s)}, \tag{3.3}$$

where

$$E(r,n,s)=\begin{cases} (\mu_1-s)(n-1)+F(1,n) & 1=r\leqslant n \\ (\mu_r-s)(n-r+1)+F(r,n) & 1<r\leqslant n \\ (\mu_r-s)(r-n)+F(r,n) & n<r<d \\ (\mu_d-s)(d-n-1)+F(d,n) & n<r=d; \end{cases}$$

$F(r,n)$ as in Lemma 3.

Proof. The argument is induction on s applied to (1.8)–(1.9), as in the proof of Theorem 1. For example, the induction step for $n<r<d$ is

$$\begin{aligned}
|\phi_{rk}^{(j-1)}(x)| &\leqslant |L_{rk}^{(j-1)}(x)| + \sum_{p=k+1}^{\mu_r} |L_{rk}^{(p-1)}(x_r)|\,|\phi_{rp}^{(j-1)}(x)| \\
&\leqslant Ch_n^{k-j}R^{F(r,n)} + \sum_{p=k+1}^{\mu_r} Ch_n^{k-p}R^{(k-p)(r-n)}h_n^{p-j}R^{(\mu_r-p)(r-n)+F(r,n)} \\
&\leqslant Ch_n^{k-j}R^{(\mu_r-k)(r-n)+F(r,n)}. \quad \blacksquare
\end{aligned}$$

Stability bounds here are more complicated than their analogs (2.5) and (2.6) from the previous section; however, it follows from Theorem 2 that

$$\sum_{r=1}^{d}\sum_{s=1}^{\mu_r} |\phi_{rs}^{(j-1)}(x)| \leqslant Ch_n^{1-j}R^{\max_r E(r,n,1)} \tag{3.4}$$

for $x_n \leqslant x \leqslant x_{n+1}$ and $h_n \leqslant 1$. If $d>2$ and the multiplicity is constant this becomes

$$\sum_{r=1}^{d}\sum_{s=1}^{\mu_r} |\phi_{rs}^{(j-1)}(x)| \leqslant Ch_n^{1-j}R^{e} \tag{3.5}$$

with $e=\max\{(n-1)[\mu(n+2)/2-1],\ (d-n-1)[\mu(d-n+2)/2-1]\}$.

As for sharpness, (3.1) can be shown sharp by considering the mesh

$$\begin{aligned}
x_i &= -1/R-1/R^2-\cdots-1/R^{n-i} && 1\leqslant i<n \\
x_n &= 0 && i=n \\
x_{i+1} &= 1+1/R+1/R^2+\cdots+1/R^{i-n} && n<i<d.
\end{aligned} \tag{3.6}$$

The argument is analogous to that for quasi-uniform meshes. Numerical experiments with this mesh indicate that the stability bound (3.5) is also sharp.

4. Applications

As a simple example of the sharpness of the stability bounds, consider the interpolating points $\langle 0, h, h(1+1/R),\ h(1+1/R+1/R^2)\rangle$ and $f(x)=x^4$. From (1.3) and (3.5) with $\mu=1$, $n=1$, $d=4$, $j=1$, we expect

$$\max_{[0,h]} |f-Q\hat{f}| \leqslant C(h^4+R^3\varepsilon) \tag{4.1}$$

for $\|f-\hat{f}\|=O(\varepsilon)$. With $\hat{f}(x_1)=f(x_1)+10^{-8}u$, u a random number from the uniform distribution on $[-1, 1]$, typical results are shown in Table I. A discrete maximum over 20 equally spaced points is used to estimate the norm; the notation $0.90-n$ means 0.90×10^{-n}. Calculations were done on an IBM 3081 with about 16 decimal digit accuracy.

It is important to point out that the sharpness of the bounds such as (4.1) depends heavily on a lack of smoothness in the perturbations. If $\hat{f}(x)=f(x)+\varepsilon g(x)$ with $g(x)$ smooth, then $f-Q\hat{f}=(1-Q)(f+\varepsilon g)-\varepsilon g$; consequently, from (1.3)

$$\|f-Q\hat{f}\| \leqslant C(1+\varepsilon)\,h^N+\varepsilon\,\|g\|$$

and a non-uniform mesh causes no difficulty.

A more interesting application arises in estimating the solutions of two-point boundary value problems by the method of collocation [3]. As an example, for a second-order differential equation a mesh $t_1<t_2<\cdots$ is chosen, and estimates for the solution and its slope are generated based on collocation over C^1-piecewise quintics. At the mesh points the errors in the solution and its first derivative are known to be bounded by CH^8, where $H=\max(t_{i+1}-t_i)$, whereas errors elsewhere are at best only $O(H^6)$. It seems reasonable to interpolate the high-order data with a 7th degree interpolating polynomial in order to maintain the $O(H^8)$ accuracy, globally. If symmetric interpolating points are chosen, i.e., $x_1=t_{i-1}$, $x_2=t_i$, $x_3=t_{i+1}$, $x_4=t_{i+2}$, each with multiplicity two, then

$$\|(f-Qf)^{(j-1)}\| \leqslant Ch^8 \leqslant CH^8$$

while from (1.4), in $[x_2, x_3]$, corresponding to $[t_i, t_{i+1}]$,

$$|(Qf-Q\hat{f})^{(j-1)}| \leqslant \Big\|\sum_r \sum_s |\phi_{rs}^{(j-1)}|\Big\}\, CH^8.$$

TABLE I

h	R	$\|f-Qf\|$	$\|f-Q\hat{f}\|$
0.1	10	0.137 − 4	0.132 − 4
	100	0.108 − 4	0.894 − 3
	1000	0.106 − 4	0.505
	10000	0.105 − 4	0.214 + 4
0.01	10	0.137 − 8	0.176 − 5
	100	0.108 − 8	0.242 − 3
	1000	0.106 − 8	0.532
	10000	0.105 − 8	0.685 + 3

For this case, the stability bound (3.5) yields ($n=2$, $d=4$, $j=1$)

$$|Qf - Q\hat{f}| \leqslant CR^3H^8$$

for locally quasi-uniform meshes. Hence, there can be a considerable degradation of accuracy for highly non-uniform meshes. This degradation is quite apparent in actual calculations, as in [3], and in light of the remarks from the previous paragraph, it says that the collocation error cannot be a smooth function.

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